Student Iwasawa Theory Seminar

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The goal of this seminar is to prepare for the 2018 Arizona Winter School by learning some Iwasawa theory: namely we'll build up to the Main Conjecture (for cyclotomic fields) and understand the objects in play and some ingredients in the proof, and we'll learn some results of Mazur on controlling ranks of elliptic curves and sizes of Selmer groups up $\mathbb{Z}_p$-extensions of number fields. An excellent five page introduction to this material is the introduction to Sharifi's notes on Iwasawa theory, available at http://math.ucla.edu/~sharifi/iwasawa.pdf. Each talk will be 1 hour long, followed by 15 minutes for questions and chatting about the talk. The seminar will run 3:45-5:00 on Tuesdays in Eckhart 206; note that number theory seminar is 2:00-3:30 and tea is at 3:30, so we're taking a 15 minute break to get tea between these two seminars.

Lecture 1: Iwasawa's Results on Class Groups.
Date: Tuesday January 9, 2018.
Speaker: Karl Schaefer.

Following Serre's article "Classes des Corps Cyclotomiques" [7], introduce the first major result in Iwasawa theory: if p^{e_n} is the size of the p-part of the class of $\mathbb{Q}(\zeta_{n+1})$, then for all $n \gg 0$, $e_n = \mu p^n + \lambda p + c$ for some integer constants μ, λ, c which are independent of n.

Set up the idea of $\mathbb{Z}_p$-extensions of number fields. Show the isomorphism between $\Lambda = \mathbb{Z}_p[\![T]\!]$ (formal power series with $\mathbb{Z}_p$-coefficients) and the completed group ring of a $\mathbb{Z}_p$-extension. Discuss without proof the classification of (finitely-generated) Λ-modules (up to quasi-isomorphism), and apply this to the module which is the inverse limit of the p-parts of the class groups of $\mathbb{Q}(\zeta_{n+1})$ to get Iwasawa's result. If time permits, say something about what is known about μ, λ, c.

References: [7].

Lecture 2: Classical results linking L-functions and class groups.
Date: Tuesday January 16, 2018.
Speaker: Mathilde Gerbelli-Gauthier.

This talk covers some classical results which Iwasawa theory generalizes greatly, the material may be found in Washington's book "Cyclotomic Fields" [8]. Recall the definition and congruence properties of Bernoulli numbers and generalized Bernoulli numbers, discussing their appearance as special values of the Riemann zeta function and Dirichlet L-functions. Using (but not proving) the analytic class number formula, prove Kummer's criterion linking regularity of a prime p and the divisibility of a Bernoulli numbers by p.

Introduce Galois actions into the above discussion, separating out the contributions of individual Dirichlet L-functions and eigenspaces of the class group in the equality provided by the analytic class number formula. Sketch the proof of Herbrand's theorem that if the ω^i-eigenspace of the class group is non-trivial, then B_{p-i} is divisible by p.

References: [8], material from Chapters 4, 5, 6.

Lecture 3: Two approaches to p-adic L-functions.
Date: Tuesday January 23, 2018.

Speaker: Zhiyuan Ding.

This talk sets up the definition of p-adic L-function in two different ways, and ends up with the statment of the main conjecture (that the two definitions give the same L-function).

Define p-adic L-functions, motivating with studying congruences of bernoulli numbers introduced last time. Katz's article [4] gives an excellent treatment of the construction of p-adic L-functions, one can also use Chapters 5 and 7 of [8].

Define characteristic ideals of finitely-generated Λ-modules.

The first evidence for the main conjecture is the following. Washington 10.16: assuming Vandiver's Conjecture, the i-th part of the class field tower has a p-adic L-function as generating its characteristic ideal. Sketch a proof of this.

End with a statement of the main conjecture.

References: the first half of [4] for p-adic L-functions, [8] for other material, [1] for material on the main conjecture.

Lecture 4: Ribet's Converse to Herbrand's Theorem.
Date: Tuesday January 30, 2018.
Speaker: Nazerke Bakytzhan.

Recall Herbrand's theorem, and state Ribet's converse. Give an overview of the proof of Ribet's converse, using the hexagon found in Mazur [5]. The steps are: Bernoulli numbers, Eisenstein Series, Modular Forms, Galois Representations, Unramified Extensions, Class Groups.

Time permitting, deduce the results of Herbrand and Ribet (in fact, even stronger versions) from the main conjecture.

References: the original article [6] is only 12 pages long, a more expanded account can be found in [5].

Lecture 5: Sketch of the Main Conjecture.
Date: Tuesday February 6, 2018.
Speaker: Drew Moore.

Restate the Main Conjecture. Deduce as many corollaries as possible from the main conjecture (see [8]). Sketch the proof, emphasizing the main techniques that are employed and how Ribet's method is generalized.

References: Washington's book [8] (the second edition) has a Chapter 15 focusing on the main conjecture, [1] is a relatively short overview of the proof of the main conjecture.

Lecture 6: Elliptic Curves I.
Date: Tuesday February 13, 2018.
Speaker: Eric Stubley.

Start with a statement of (one of) the theorem(s) we're working towards: if E is an elliptic curve over a number field F and $F_\infty = \cup_n F_n$ is a $\mathbb{Z}_p$-extension of F (i.e. $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z}$, $\mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$), then (under some assumptions) the ranks $\mathrm{rank}_{\mathbb{Z}}(E(F_n))$ are bounded for $n \geq 0$. Explain some relationships between studying ranks of curves in $\mathbb{Z}_p$-extensions to BSD, relate BSD to analytic class number formula which was used in the cyclotomic main conjecture ([3] has some good discussion of this in the introduction).

Speed through some background on elliptic curves (Chapter 1 of [2]). Focus on defining things from Chapter 2 carefully, spending lots of time getting used to working with p-primary groups like $\mathbb{Q}_p/\mathbb{Z}_p$ which take a while to get used to (Chapter 2 of [2]). Be sure to include the Galois cohomology proof of the Weak Mordell-Weil theorem. The rest of Chapter 2 is devoted to understanding the image of the local Kummer maps for elliptic curves, so make sure these results are stated.

References: material for this lecture and the next two is mostly from [2], see also [3] for some context.

Lecture 7: Elliptic Curves II.
Date: Tuesday February 20, 2018.
Speaker: Shiva Chidambaram.

Chapter 3 of [2] is on Λ-modules, so we'll have seen much of the material here already. To work through the proof of Mazur's theorem in Chapter 4 we may need some specific results from Chapter 3 (likely some form of a p-adic Weierstrass preparation theorem), so make sure that is covered. Start working through Chapter 4.

References: [2].

Lecture 8: Elliptic Curves III.
Date: Tuesday February 27, 2018.
Speaker: Yiwen Zhou.

Picking up where the previous talk left off, finish the content of Chapter 4 of [2]. The main result is theorem 4.1, studying behaviour of Selmer groups up a $\mathbb{Z}_p$-extension. Use this result to deduce theorem 1.6, which is the boundedness result of the ranks of $E(F_n)$ which we're aiming for.

References: [2].

References

[1] John Coates. "The work of Mazur and Wiles on cyclotomic fields". In: *Bourbaki Seminar, Vol. 1980/81*. Vol. 901. Lecture Notes in Math. Springer, Berlin-New York, 1981, pp. 220–242.

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[8] Lawrence C. Washington. *Introduction to cyclotomic fields*. Second. Vol. 83. Graduate Texts in Mathematics. Springer-Verlag, New York, 1997, pp. xiv+487. ISBN: 0-387-94762-0. URL: https://doi.org/10.1007/978-1-4612-1934-7.